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On the Determination of the Ternary Modular Groups.

By R. L. Börger.

Introduction.

The subgroups of the linear fractional group $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$ in $GF[p^n]$ have been determined.* Closely related to this is the determination of the binary groups of determinant unity in the $GF[p^n]$.† The subgroups of the special ternary group $GF[p]$ have been found.‡ The object of the present paper is to apply the method of the paper last cited to the group in $GF[p^2]$. The order of the special ternary linear group in $GF[p^2]$ is $p^6(p^6-1)(p^4-1)$. The subgroups of order a multiple of p, $p^i M (i = 6, 5, 4)$ (M prime to p) are determined in this paper. The determination of the groups of these high orders applies in particular to the question of the minimum index of a subgroup of total group. For $i = 3, 2, 1$ the method can not be employed and recourse must be had to special devices.

Subgroups of Order a Power of p.

1. The order of the special linear ternary group G in the $GF[p^2]$ is $p^6(p^6-1)(p^4-1)$. Within G every subgroup of order a power of p is conjugate with a subgroup of the group G_{p^6} of the operations

$$[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{pmatrix} \qquad (1)$$

with the α_{ij} arbitrary in the $GF[p^2]$.

Since§ the commutator of G_{p^6} is of order p^2 and every operation is of order p, $(p \neq 2)$, there are $\frac{p^4-1}{p-1}$ subgroups of order p^5. They are given by a linear relation modulo p,

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* Gierster, Wiman; Moore, *University of Chicago Decennial Publications*, Vol. IX (1903), pp. 141–190; Dickson: "Linear Groups."

† Made by Dickson — unpublished.

‡ Dickson, American Journal, Vol. XXVII, pp. 189–202.

§ The results in §§ 1–3 are proved in detail in American Journal, Vol. XXVII, pp. 289–293.

$$l\,(a_{210},\ a_{211},\ a_{320},\ a_{321}) \equiv 0. \tag{2}$$

2. The commutator subgroup K of any group L defined by a relation (2) is of order p^2.

THEOREM: Any group L_{p^5} defined by one linear relation (2) has exactly p^2+p+1 subgroups of order p^4, defined by two independent relations (2). The number of distinct subgroups of order p^4 of G_{p^6} is thus

$$p^4+p^3+2p^2+p+1.$$

They are as follows:

(I$_a$) $a_{210} = g\,a_{320} + h\,a_{321},\quad a_{211} = l\,a_{320} + m\,a_{321}\qquad [\alpha_{21} = t\,\alpha_{32}]$.

(I) $a_{210} = g\,a_{320} + h\,a_{321},\quad a_{211} = l\,a_{320} + m\,a_{321}\qquad [\alpha_{21} \neq t\,\alpha_{32}]$.

(II) $a_{321} = h\,a_{320},\quad a_{211} = l\,a_{210} + m\,a_{320}$.

(III) $a_{321} = h\,a_{320},\quad a_{210} = m\,a_{320}$.

(IV) $a_{320} \equiv 0,\quad a_{211} = l\,a_{211} + m\,a_{321}$.

(V) $a_{320} \equiv 0,\quad a_{210} = m\,a_{321}$.

(VI) $a_{320} \equiv 0,\quad a_{321} \equiv 0$.

THEOREM. The only commutative subgroups of order p^4 are those defined by $\{\alpha_{32} = 0\}$ and $\{\alpha_{21} = t\,\alpha_{32}\}$. For the remaining subgroups of order p^4 the commutator subgroup is of order p.

3. THEOREM. For $p > 2$ the subgroups G_{p^3} of the G_{p^4}, given in the first column, are obtained by annexing to the two relations defining G_{p^4} an independent linear homogeneous relation between the elements given in the second column.

G_{p^4}. ELEMENTS.

(I$_a$) $a_{320},\quad a_{321},\quad a_{310} - \tfrac{1}{2}g\,a_{320}^2 - h\,a_{320}\,a_{321} - \tfrac{1}{2}(\nu\,g + \mu\,h)\,a_{320}^2,$

$\qquad a_{311} - \frac{h}{2\nu}a_{321}^2 - \left(g + \mu\frac{h}{\nu}\right)a_{320}\,a_{321} - \tfrac{1}{2}\left(h + \frac{\mu^2 h}{\nu} + \mu\,g\right)a_{321}^2$.

(I) $a_{320},\quad a_{321},\quad (m - g - l\mu)\,a_{310} - (h - \nu\,l)\,a_{311} - u\,a_{320}^2 - v\,a_{320}\,a_{321} - w\,a_{321}^2$.

(II) $a_{320},\quad a_{210},\quad (l + h + l\,h\,\mu)\,a_{310} - (1 + h\,l\,\nu)\,a_{311} + \tfrac{1}{2}m\,(1 + h\,\mu - h^2\nu)\,a_{320}^2$.

(III) $a_{320},\quad a_{211},\quad (1 + h\,\mu)\,a_{310} - h\,\nu\,a_{311} - \frac{m}{2}(1 + h\,\mu - h^2\nu)\,a_{320}^2$.

(IV) $a_{321},\quad a_{210},\quad (1 + l\,\mu)\,a_{310} - l\,\nu\,a_{311} - \tfrac{1}{2}m\,\nu\,a_{321}^2$.

(V) $a_{321},\quad a_{211},\quad \mu\,a_{310} - \nu\,a_{311} + \tfrac{1}{2}m\,\nu\,a_{321}^2$.

(VI) $a_{210},\quad a_{211},\quad a_{310},\quad a_{311}$.

The u, v and w used in (I) denote

$$u \equiv \tfrac{1}{2} g (m - g - l\mu) - \tfrac{1}{2} l (h - \nu l),$$
$$v \equiv m \nu l - h g - h l \mu,$$
$$w \equiv \tfrac{1}{2} m \nu (m - g - l\mu) - \tfrac{1}{2} (h - l\nu)(h + m\mu).$$

The Galois Field is defined by the irreducible congruence $\rho^2 \equiv \mu\rho + \nu \pmod{p}$.

4. The subgroups of order p^2 of any abelian group G_{p^3} represented in the first column are given by annexing, to the relations defining the G_{p^3}, an independent linear homogeneous relation between the elements of the second column.

G_{p^3}	ELEMENTS.
$(\mathrm{I_a})$	$a_{320}, \quad a_{321}, \quad a_{310} - \tfrac{1}{2} g a_{320}^2 - h a_{320} a_{321} - \tfrac{1}{2}(\nu g + \mu h) a_{321}^2,$ $a_{311} - \dfrac{h}{2\nu} a_{321}^2 - \left(g + \dfrac{\mu h}{\nu}\right) a_{320} a_{321} - \tfrac{1}{2}\left(h + \dfrac{\mu^2 h}{\nu} + \mu g\right) a_{321}^2.$
$\mathrm{I}_{1,3}$	$a_{320}, \quad a_{310} - \tfrac{1}{2} g a_{320}^2, \quad a_{311} - \tfrac{1}{2} l a_{320}^2.$
$\mathrm{I}_{2,3}$	$a_{321}, \quad a_{310} - \dfrac{\nu h}{2} a_{321}^2, \quad a_{311} - \tfrac{1}{2}(h + \mu m) a_{321}^2.$
I_3	$a_{320}, \quad a_{310} - \tfrac{1}{2}\lambda a_{320}^2, \quad a_{311} - \tfrac{1}{2}\eta a_{320}^2.$
$\mathrm{II}_{1,3}$	$a_{320}, \quad a_{310} - \tfrac{1}{2}\nu h m a_{320}^2, \quad a_{311} - \tfrac{1}{2}(m + \mu h m) a_{320}^2.$
$\mathrm{II}_{2,3}$	$a_{210}, \quad a_{310}, \quad a_{311}.$
II_3	$a_{320}, \quad a_{310} - \tfrac{1}{2}[k + \nu h(lk + m)] a_{320}^2, \quad a_{311} - \tfrac{1}{2}\{lk + m + hk + \mu h(lk + m)\} a_{320}^2.$
$\mathrm{III}_{1,3}$	$a_{320}, \quad a_{310} - \tfrac{1}{2} m a_{320}^2, \quad a_{311} - \dfrac{hm}{2} a_{320}^2.$
$\mathrm{III}_{2,3}$	$a_{211}, \quad a_{310}, \quad a_{311}.$
III_3	$a_{320}, \quad a_{311} - \tfrac{1}{2}(m + \nu h k) a_{320}^2, \quad a_{311} - \tfrac{1}{2}(k + h m + \mu h k) a_{320}^2.$
$\mathrm{IV}_{1,3}$	$a_{321}, \quad a_{310} - \tfrac{1}{2} m \nu a_{321}^2, \quad a_{311} - \tfrac{1}{2} m \nu a_{321}^2.$
$\mathrm{IV}_{2,3}$	$a_{210}, \quad a_{310}, \quad a_{311}.$
IV_3	$a_{321}, \quad a_{310} - \dfrac{\nu}{2}(k l + m) a_{321}^2, \quad a_{311} - \dfrac{m\nu}{2} a_{321}^2.$
$\mathrm{V}_{1,3}$	$a_{321}, \quad a_{310}, \quad a_{311} - \tfrac{1}{2} m a_{321}^2.$
$\mathrm{V}_{2,3}$	$a_{211}, \quad a_{310}, \quad a_{311}.$
V_3	$a_{321}, \quad a_{310} - \nu \dfrac{k}{2} a_{321}^2, \quad a_{311} - \tfrac{1}{2}(m + m k) a_{321}^2.$
VI	$a_{210}, \quad a_{211}, \quad a_{310}, \quad a_{311}.$

$$[\lambda \equiv g + hk + \nu k l + \nu k^2 m, \quad \eta \equiv l + mk + gk + hk^2 + \mu k l + \mu k^2 m,$$
$$k = -l_1/l_2.]$$

The notation in first column meaning that coefficients corresponding to the subscripts are 0, the others being destinct from 0. Thus: $\mathrm{V}_{1,3}$ means that linear relation defining the G_{p^3} has $l_1 = l_3 = 0$, $l_2 \neq 0$; V_3 means that linear relation defining the G_{p^3} has $l_3 = 0$, $l_1 \neq 0$, $l_2 \neq 0$.

The remaining subgroups of order p^3 are non-abelian.

Lemma.[*] The commutator subgroup of any non-abelian group of order p^3 is of order p.

Proof: If G_p is a subgroup of invariant operations, the quotient group G_{p^3}/G_p is of order p^2. Hence G_p contains the commutator subgroup of G_{p^3}. Since G_{p^3} is non-abelian the commutator is G_p.

It follows that the number of subgroups of order p^2 of G_{p^3} (G_{p^3} being non-abelian and operations all of order p) is $p+1$. They are given by annexing to the relations defining G_{p^3} an independent linear homogeneous relation between the elements of the first two columns in § 4.

The subgroups of order p are easily found. They will not be written here.

Subgroups of Order a Multiple of p.

5. *Lemma* I. Every operation (β_{ij}) permutable with a subgroup, not the identity, of G_{p^6} is contained in the set:

$$\begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \quad \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \quad \begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}. \tag{3}$$

Proof: Among the conditions for $(\beta_{ij})\,[\alpha] = [\gamma]\,(\beta_{ij})$ are:

$$\beta_{11} + \beta_{12}\alpha_{21} + \beta_{13}\alpha_{31} = \beta_{11}, \tag{1}$$

$$\beta_{21} + \beta_{22}\alpha_{21} + \beta_{23}\alpha_{31} = \beta_{11}\gamma_{21} + \beta_{21}, \tag{2}$$

$$\beta_{12} + \beta_{13}\alpha_{32} = \beta_{12}, \tag{3}$$

$$\beta_{22} + \beta_{23}\alpha_{32} = \beta_{12}\gamma_{21} + \beta_{22}, \tag{4}$$

$$\beta_{32} + \beta_{33}\alpha_{32} = \beta_{12}\gamma_{31} + \beta_{22}\gamma_{32} + \beta_{32}, \tag{5}$$

$$\beta_{23} = \beta_{23} + \gamma_{21}\beta_{13}. \tag{6}$$

If $\alpha_{21} \equiv 0$, $\alpha_{32} \not\equiv 0$, $\alpha_{31} \not\equiv 0$, then $\beta_{13} = \beta_{23} = 0$ by (1) and (2).

In like manner we treat the remaining cases and prove the lemma.

Lemma II. The only factors $\equiv 1 \pmod{p^4}$ of $(p^6-1)(p^4-1) \equiv \omega$ are 1 and ω. The proof is similar to that in the paper[†] referred to and will not be given here.

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[*] Miller, *Bulletin American Mathematical Society*, 1897–98, p. 137.

[†] Dickson, AMERICAN JOURNAL, Vol. XXVII, p. 191.

Lemma III. Any binary transformation $B=\begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}$, $\gamma \neq 0$, and all the $E_\lambda=\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix}$ generate every binary transformation T of determinant unity.

For proof, see reference above.

Lemma IV. If a group G_1, of order $p^\beta N$ (N prime to p), contains a subgroup of G_{p^α} ($\alpha > \beta$) of order p^β and if P is an operation of G_1 permutable with G_{p^α}, then P is permutable with G_{p^β}.

Proof: Suppose P were not permutable with G_{p^β}. It would transform G_{p^β} into G'_{p^β}; and $\{G_{p^\beta}, G'_{p^\beta}\}$, being of order a power of p, would be of order $p^{2\beta-c}$ (where p^c is the order of the greatest subgroup common to G_{p^β} and G'_{p^β}). But for all values of $c<\beta$ we have $p^{2\beta-c}>p^\beta$. Hence $c=\beta$ and $G'_{p^\beta}\equiv G_{p^\beta}$.

Lemma V. The operations permutable with every group G_{p^3} defined by

$$\alpha_{21}=0, \quad k\,a_{321}+h\,a_{320}=0$$

are contained in the set (3_1).

Proof: By Lemma I, they are contained in (3_2). Transforming G_{p^3} by (β_{ij}) $(\beta_{13}=\beta_{23}=0)$ it becomes $G'_{p^3}[\gamma_{ij}]$, having

$$\gamma_{31}\equiv\frac{\beta_{11}}{\beta_{33}}\alpha_{31}+\frac{\beta_{21}}{\beta_{33}}\alpha_{32}, \qquad \gamma_{32}\equiv\frac{\beta_{12}}{\beta_{33}}\alpha_{31}+\frac{\beta_{22}}{\beta_{33}}\alpha_{32}.$$

Call

$$\frac{\beta_{11}}{\beta_{33}}\equiv c_{11}+\rho\,d_{11}, \quad \frac{\beta_{22}}{\beta_{23}}\equiv c_{22}+\rho\,d_{22},$$
$$\frac{\beta_{12}}{\beta_{33}}\equiv c_{12}+\rho\,d_{12}, \quad \frac{\beta_{21}}{\beta_{22}}\equiv c_{21}+\rho\,d_{21}.$$

Then

$$\gamma_{31}\equiv(c_{12}\,a_{310}+c_{22}\,a_{320}+\nu\,d_{12}\,a_{311}+\nu\,d_{22}\,a_{321})$$
$$+\rho\,(d_{12}\,a_{310}+c_{12}\,a_{311}+\mu\,d_{12}\,a_{311}+\mu\,d_{22}\,a_{321}+c_{22}\,a_{321}+d_{22}\,a_{320}).$$

Setting $\alpha_{32}=0$ and recalling the definition of G_{p^3}, we get the relation:

$$(c_{12}h+d_{12}k)\,a_{310}+(\nu\,d_{12}h+c_{12}k+\mu\,d_{12}k)\,a_{311}=0.$$

Since a_{310} and a_{311} are arbitrary, we have

$$c_{12}h+d_{12}k=0, \quad c_{12}k+d_{12}(\nu\,h+\mu\,k)=0.$$

Thus $c_{12}=d_{12}=0$, since the determinant of their coefficients equals $\nu h^2+\mu h k-k^2$, which vanishes only for the excluded case $h=k=0$, since $\rho^2\equiv\mu\rho+\nu$ was assumed irreducible.

6. Let H be a subgroup of order p^6N (N prime to p) normalized to contain G_{p^6}. Then, by methods similar to those employed in the paper referred to, we derive the following

THEOREM. *Within G every subgroup H of order a multiple of p^6 is conjugate with one of the following:*

a) *The group of all the $p^6 f.g$ operations (3_1) with $\alpha_{11}^f = 1$, $\alpha_{22}^g = 1$, $\alpha_{33} = \alpha_{11}^{-1} \alpha_{22}^{-1}$.*

b) *The group of all the $f p^6 (p^4 - 1)$ operations (3_2) with $\alpha_{33}^f = 1$, $\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} = \alpha_{33}^{-1}$.*

c) *The group of all the $f p^6 (p^4 - 1)$ operations (3_3) with $\alpha_{11}^f = 1$, $\alpha_{22} \alpha_{33} - \alpha_{23} \alpha_{32} = \alpha_{11}^{-1}$, where f and g are any divisors of $p^2 - 1$.*

7. Let H be a subgroup of G of order $p^5 N$ (N prime to p) normalized to contain a subgroup G_{p^5} of G_{p^6}. Let the G_{p^5} be defined by

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{321} = 0.$$

(I). We exclude for the present the special case $l_1 = l_2 = 0$ and the special case $l_3 = l_4 = 0$ (treated under II and III below), and treat here the general case.

The number of conjugates to G_{p^5} is $1 + kp$, $1 + kp^2$, $1 + kp^3$, or $1 + kp^4$ (k being prime to p). We treat these cases in turn.

(I_1). Let the number of conjugates to G_{p^5} be $1 + kp$. Then two groups of order p^5 must contain a common subgroup of order p^4 * and, by a theorem † due to Frobenius and Burnside independently, H must contain an operation P permutable with G_{p^4} but not with G_{p^5}. Since neither $l_1 = 0$, $l_2 = 0$ nor $l_3 = 0$, $l_4 = 0$ in the case before us, this common G_{p^4} must have neither $\alpha_{21} \equiv 0$ nor $\alpha_{32} \equiv 0$. Therefore the operations permutable with G_{p^4} are contained in (3_1) and P is also permutable with G_{p^6}. Hence P must, by Lemma IV, be permutable with G_{p^5}, in contradiction with the Frobenius-Burnside Theorem. Hence the number of conjugates to G_{p^5} can not be $1 + kp$.

(I_2). Suppose the number of conjugates to G_{p^5} is $1 + kp^2$. The case in which $\alpha_{21} \not\equiv 0$ in the common G_{p^3} has been covered by the previous argument.

Let next $\alpha_{21} \equiv 0$ in the common G_{p^3}. The groups of order p^3 are then defined by

$$\left.\begin{array}{l} a_{210} = a_{211} = 0 \\ a_{321} = h\, a_{320} \end{array}\right\}, \quad \left.\begin{array}{l} a_{210} = a_{211} = 0 \\ a_{320} = 0 \end{array}\right\}. \tag{5}$$

But for either of these groups, the operation P is contained in (3_1), and as before we conclude that the number of conjugates can not be $1 + kp^2$.

(I_3). Let the number of conjugates be $1 + kp^3$. For the common G_{p^2}, we have the following possibilities:

a) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \not\equiv 0$; b) $\alpha_{21} \equiv 0$, $\alpha_{32} \not\equiv 0$; c) $\alpha_{21} \equiv 0$, $\alpha_{32} \equiv 0$; d) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \equiv 0$.

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* Burnside: "Theory of Groups," p. 94, Cor. II.

† *Op. cit.*, p. 97.

The operation P for the G_{p^2} defined in either a) or c) is contained in (3_1) and is therefore permutable with G_{p^5}. In b) we note that G_{p^5} must contain a subgroup G_{p^3} defined by (5_1) or (5_2). The operation permutable with the G_{p^2} in b) is contained either in (3_1) or (3_2), and we need only consider the case in which P is in (3_2). Since under this hypothesis P is not permutable with either of the groups G_{p^3} defined in (5_1), (5_2), it will transform either of them into a new G'_{p^3} and $\{G_{p^3} G'_{p^3}\}$ is of order p^4 with $\alpha_{21} \equiv 0$. This G_{p^4} is not a subgroup of G_{p^5}, but is permutable with it. Hence $\{G_{p^5}, G_4\}$ must be of order p^6, in contradiction with the hypothesis on the order of H. Hence P must be contained in (3_1). It will then be permutable with G_{p^5}, contradicting the Frobenius-Burnside Theorem. A similar argument applies to the case d). We now conclude that the number of conjugates to G_{p^5} can not be $1 + kp^3$.

(I_4). By Lemma II the only divisor of the order of G of the form $1 + kp^4$ is $(p^6 - 1)(p^4 - 1)$. If the number of conjugates to G_{p^5} were $1 + kp^4$, the index of H under G would be p, whereas the order of $LF\{3, p^2\}$ is not a divisor of $p!$.

(II). Let H contain a G_{p^5} defined by $l_1 a_{210} + l_2 a_{211} = 0$. We shall prove that such a group H can not contain any operation (β_{ij}) not contained in (3_2). We first show that if any operation (β_{ij}) has β_{13}, β_{23} not both 0, then H contains an operation (γ_{ij}) having $\gamma_{12} = 0$, $\gamma_{13} = 0$, $\gamma_{23} \neq 0$.

a) If $\beta_{12} = \beta_{13} = 0$, while $\beta_{23} \neq 0$, this (β_{ij}) is such an operation.

b) The case $\beta_{12} \neq 0$, $\beta_{13} = 0$, may be disposed of by replacing (β_{ij}) by its inverse. We need then consider only

c) $\beta_{13} \neq 0$. Transforming G_{p^5} by β_{ij} we get (γ_{ij}), where

$$\gamma_{11} \equiv \bar\beta_{12}\beta_{11}\alpha_{21} + \bar\beta_{13}\beta_{11}\alpha_{31} + \bar\beta_{13}\beta_{21}\alpha_{32} + 1, \tag{1}$$

$$\gamma_{12} \equiv \bar\beta_{12}\beta_{12}\alpha_{21} + \bar\beta_{13}\beta_{12}\alpha_{31} + \bar\beta_{13}\beta_{22}\alpha_{32}, \tag{2}$$

$$\gamma_{13} \equiv \bar\beta_{12}\beta_{13}\alpha_{21} + \bar\beta_{13}\beta_{13}\alpha_{31} + \bar\beta_{13}\beta_{23}\alpha_{32}, \tag{3}$$

$$\gamma_{21} \equiv \bar\beta_{22}\beta_{11}\alpha_{21} + \bar\beta_{23}\beta_{11}\alpha_{31} + \bar\beta_{23}\beta_{21}\alpha_{32}, \tag{4}$$

$$\gamma_{22} \equiv \bar\beta_{22}\beta_{12}\alpha_{21} + \bar\beta_{23}\beta_{12}\alpha_{31} + \bar\beta_{23}\beta_{22}\alpha_{32} + 1, \tag{5}$$

$$\gamma_{23} \equiv \bar\beta_{22}\beta_{13}\alpha_{21} + \bar\beta_{23}\beta_{13}\alpha_{31} + \bar\beta_{23}\beta_{23}\alpha_{32}, \tag{6}$$

$$\gamma_{31} \equiv \bar\beta_{32}\beta_{11}\alpha_{21} + \bar\beta_{33}\beta_{11}\alpha_{31} + \bar\beta_{33}\beta_{21}\alpha_{32}, \tag{7}$$

$$\gamma_{32} \equiv \bar\beta_{32}\beta_{12}\alpha_{21} + \bar\beta_{33}\beta_{12}\alpha_{31} + \bar\beta_{33}\beta_{22}\alpha_{32}, \tag{8}$$

$$\gamma_{33} \equiv \bar\beta_{32}\beta_{13}\alpha_{21} + \bar\beta_{33}\beta_{13}\alpha_{31} + \bar\beta_{33}\beta_{23}\alpha_{32} + 1, \tag{9}$$

where $\bar\beta_{ij}$ is the adjoint minor of β_{ji} in $|\beta_{ji}|$.

We wish to show that we can make $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{23} \neq 0$ by properly choosing $\alpha_{21}, \alpha_{31}, \alpha_{32}$. We can by a normalization of the operator (β_{ij}) make $\bar\beta_{12} \neq 0$ without disturbing β_{13}; viz., by multiplying (β_{ij}) on the left by $B_{32\lambda}$. Now suppose that $\gamma_{23} = 0$. Then, setting $\alpha_{32} = 0$, we have the following equations:

$$\bar\beta_{22}\alpha_{21} + \bar\beta_{23}\alpha_{31} = 0, \tag{6'}$$

$$\bar\beta_{32}\alpha_{21} + \bar\beta_{33}\alpha_{31} = 0, \tag{9'}$$

$$\bar\beta_{12}\alpha_{21} + \beta_{13}\alpha_{31} = 0. \tag{3'}$$

Multiplying (6′) by β_{32}, (9′) by β_{33} and (3′) by β_{31}, adding and applying $\Delta = 1$, we get $\alpha_{31} = 0$. This is impossible if $\bar\beta_{12} \neq 0$. It follows, then, that the operation (γ_{ij}) fulfils the required conditions if $\bar\beta_{13} \neq 0$. If $\bar\beta_{13} = 0$ we can set $\alpha_{32} = 0$ as before, and now put $\alpha_{21} = 0$, thus making $\gamma_{12} = \gamma_{13} = 0$. Then, if $\gamma_{23} = 0$, we must have equations (6′), (9′), (3′) true for arbitrary values of α_{31}. From the previous argument these equations can only be simultaneously true when $\alpha_{31} = 0$. Therefore the operation (γ_{ij}) always has $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{23} \neq 0$.

Transforming $[\alpha]$ by (γ_{ij}) we get an operation

$$\begin{pmatrix} 1 & 0 & 0 \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}, \qquad \begin{aligned} \delta_{21} &\equiv \bar\gamma_{22}\alpha_{21} + \bar\gamma_{23}\alpha_{31} + \bar\gamma_{23}\gamma_{21}\alpha_{32}, \\ \delta_{31} &\equiv \bar\gamma_{32}\alpha_{21} + \bar\gamma_{33}\alpha_{31} + \bar\gamma_{33}\gamma_{21}\alpha_{32}. \end{aligned}$$

Setting $\alpha_{21} = 0$, $\alpha_{31} = -\gamma_{21}\alpha_{32}$, then $\delta_{21} = \delta_{31} = 0$ and $\delta_{23} = \gamma_{23}^2\alpha_{32} \neq 0$ if $\alpha_{32} \neq 0$.

Therefore, by Lemma III, the group H contains every binary transformation of determinant unity on ξ_2, ξ_3. Hence it contains

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \qquad (\mu \text{ arbitrary}),$$

and this, together with $B_{31,\lambda}\, B_{32,\delta}$, generates a group of order p^6. But this contradicts the hypothesis on the order of H.

(III). By a reflexion[*] on the left diagonal we may draw the conclusion, with reference to the group G_{p^5} defined by

$$l_3\,a_{320} + l_4\,a_{321} = 0,$$

that it can not contain any operation (β_{ij}) not contained in (3_3). H is then multiply isomorphic with the group of binary substitutions on ξ_1, ξ_2 of order pt (t prime to p).

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[*] Dickson, American Journal, Vol. XXVIII, p. 9.

We have now established the following theorem covering the results of the three cases I, II, III.

THEOREM. *Any subgroup* H *of order* $p^5 N$ *of the linear ternary homogeneous group* G *defined by the relation*

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{321} = 0$$

(*in which neither* $l_1 = l_2 = 0$ *nor* $l_3 = l_4 = 0$) *contains a self-conjugate subgroup* G_{p^5}, *and within* G H *is conjugate with a group of operations given by* (3_1) *having* $\alpha_{22}^3 = 1$, $\alpha_{11}\alpha_{22}\alpha_{33} = 1$.

If H *contains a subgroup* G_{p^5} *defined by the relation* $l_1 a_{210} + l_2 a_{211} = 0$ or $l_3 a_{320} + l_4 a_{321} = 0$, H *is conjugate within* G *with a group of operations of the form* (3_1) or (3_2).

Subgroups of Order a Multiple of p^4.

8. Let H be a subgroup of order $p^4 N$ normalized to contain a subgroup G_{p^4} of G_{p^6}. First, let G_{p^4} be the special group $\{\alpha_{21} = 0\}$. We shall prove that it is self-conjugate under H.

If the number of conjugates to G_{p^4} were $1 + kp$ (k prime to p), then H would contain an operation P permutable with the G_{p^3} common to two G_{p^4} of the conjugate set to which G_{p^4} belongs, but not permutable with G_{p^4}, according to the Frobenius-Burnside Theorem. The operations permutable with the common G_{p^3} are, however, permutable with G_{p^4}. In the same way it follows that there can not be $1 + kp^2$ or $1 + kp^3$ conjugates to G_{p^4}. Finally, there can be no group H containing a G_{p^4} having $1 + kp^4$ conjugates. Indeed, by Lemma II, such a group H would be of index p^2 under G. But the linear homogeneous group G is isomorphic with the linear fractional $LF(3, p^2)$ and, if such an H existed, $LF(3, p^2)$ would be representable as a substitution group on p^2 letters and also its subgroup the linear fractional $LF(2, p^2)$. The latter can not be represented as a substitution on p^2 letters except when $p = 3$ (references in the Introduction). No such subgroup H can exist when $p = 3$, for the order of G when $p = 3$ is not a divisor of $p^2!$. Hence the subgroup $\{\alpha_{21} = 0\}$ is self-conjugate in H.

Next, by a reflexion on the left diagonal we reach the same conclusion with reference to $\{\alpha_{32} \equiv 0\}$.

Let H now be normalized to contain any one of the remaining groups of order p^4. If the number of conjugates to G_{p^4} is $1+kp^i$ (k prime to p, $i=1,2,3$), H must* contain an operation P permutable with $G_{p^{4-i}}$ but not with G_{p^4}.

First, let $i=1$. Then the greatest common subgroup is of order p^3, and the operation P is contained in one of the sets (3_1), (3_2), (3_3). We shall not consider (3_3), for the results obtained in case P lies in (3_2) are immediately applicable to the case when P lies in (3_3) by reflexion on the left diagonal. With this agreement the greatest common G_{p^3} of two groups of order p^4 is of one of two types:

$$\text{a) } \alpha_{21} \not\equiv 0, \quad \alpha_{32} \not\equiv 0;$$
$$\text{b) } \alpha_{21} \equiv 0, \quad \alpha_{32} \not\equiv 0.$$

The operations permutable with either of the groups a), b) are contained in the set (3_1) by Lemmas II and V, and P therefore is permutable with G_{p^4}, contradicting the Frobenius-Burnside Theorem. The number of conjugates to G_{p^4} under H can not be $1+kp$.

Next, let $i=2$ or 3. Then G_{p^m}, the greatest common subgroup, will be of order p^2 or p. P can not be contained in (3_1), since that leads to a contradiction of the Frobenius-Burnside Theorem. We consider P in the set (3_2). Any G_{p^4} of the set $(p, 3)$ contains the subgroup G_{p^2} defined by $\alpha_{21}\equiv\alpha_{32}\equiv 0$. The operations permutable with this G_{p^2} are contained in (3_1). Hence P in (3_2) is not permutable with G_{p^2}. It transforms G_{p^2} into G'_{p^2}; and $\{G_{p^2}, G'_{p^2}\}$ is of order p^4, G_{p^4} having $\alpha_{21}\equiv 0$. Hence $\{G'_{p^4} G_{p^4}\}$ is of order a power of p higher than p^4, and this contradicts the hypothesis on H. Hence the number of conjugates to G_{p^4} can not be $1+kp^2$ or $1+kp^3$. We have previously shown that the number can not be $1+kp^4$. Hence G_{p^4} must be self-conjugate under H, and we may state the following

THEOREM. *Every subgroup H, of order a multiple of p^4, of G contains a self-conjugate group G_{p^4}. Within G, H is conjugate with a group of operations* (3_2) *or a group of operations* (3_3).

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* Burnside: "Theory of Groups," p. 94, Cor. II, and the theorem on p. 97.